Introduction to Mathematics and Modeling

lecture 2

Exponentials and logarithms

UNIVERSITY OF TWENTE.

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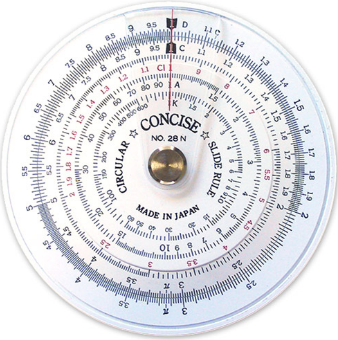
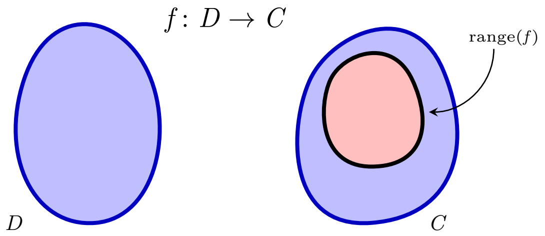
Part 2

This week



Model 28 circular slide rule by Concise Ltd.

1. Section 1.5: exponential functions
2. Section 1.6: inverse functions and logarithms



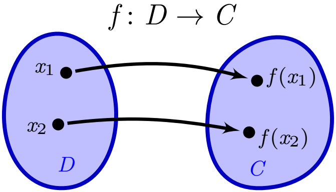
- Exactly one arrow departs from every point in D.
- Points in C that are not in the range of f are not hit by an arrow.
- Points in the range of f may be hit by more than two arrows.

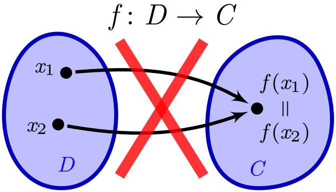
Observation

If we reverse the direction of the arrows, then the result might not be a function.

Definition

A function $f\colon D \to C$ is **one-to-one** *if $f(x_1) \neq f(x_2)$ for every x_1 and $x_2 \in D$ with $x_1 \neq x_2$.*

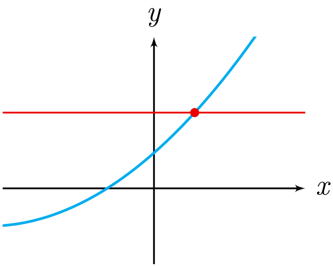




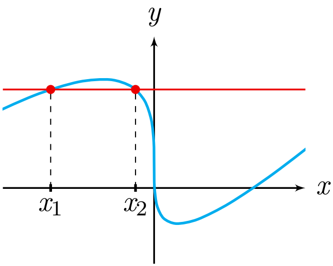
- This is equivalent with: for all x_1 and $x_2 \in D$ we have: if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
- For a one-to-one function every point in C is the end point of *at most* one arrow.

Example

The function $f(x) = 2x - 1$ *is one-to-one.*



one-to-one



not one-to-one

The Horizontal line Test

If f is one-to-one, then a horizontal line intersects the graph of f in *at most* one point.

Example

The function $f(x) = 2x^2 - 1$ is not one-to-one.

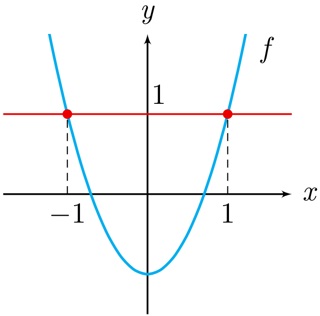
- Notice that from $f(x_1) = f(x_2)$ follows: $x_1^2 = x_2^2$, which does *not* imply $x_1 = x_2$.
- Observe that

$$f(1) = 2 \cdot 1^2 - 1 = 1,$$

and

$$f(-1) = 2 \cdot (-1)^2 - 1 = 1,$$

hence $f(1) = f(-1)$.

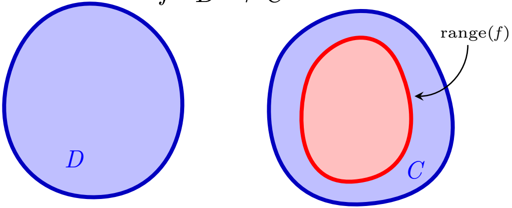


- The graph of f does *not* satisfy the horizontal line test.
- One counterexample suffices.

Theorem

If $f\colon D \to C$ is one-to-one, then reversing the arrows yields a function from the range of f to D.

$f\colon D \to C$



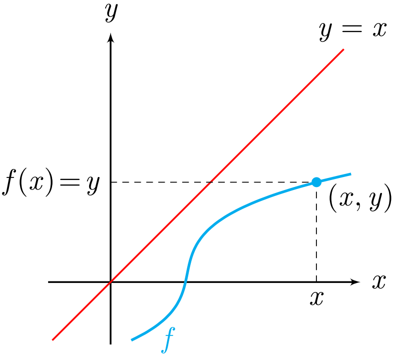
$f^{-1}\colon \mathrm{range}(f) \to D$

- This function is called the **inverse of** f, and is denoted as f^{-1}.

- If $y = f(x)$, then $x = f^{-1}(y)$.
- Finding the inverse means: solve the equation $y = f(x)$ for x.

Example

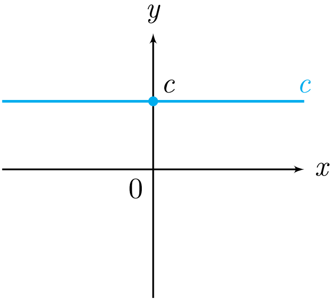
Find the inverse of $f(x) = 2x - 1$.



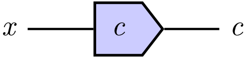
- Let $y = f(x)$. Then (x, y) lies on the graph of f.
- From $y = f(x)$ follows $x = f^{-1}(y)$, so (y, x) lies on the graph of f^{-1}.
- The points (x, y) and (y, x) are reflected across the line $y = x$.
- The graph of f^{-1} and the graph of f are symmetric with respect to the line $y = x$.

Definition

The **constant function** $c\colon \mathbb{R} \to \mathbb{R}$ *assigns* c *to every* $x \in \mathbb{R}$.

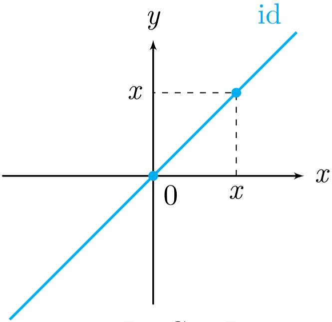


$D = \mathbb{R},\ C = \mathbb{R}$

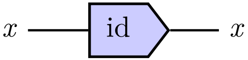


Definition

The **identical map** $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}$ *assigns* x *to every* $x \in \mathbb{R}$.



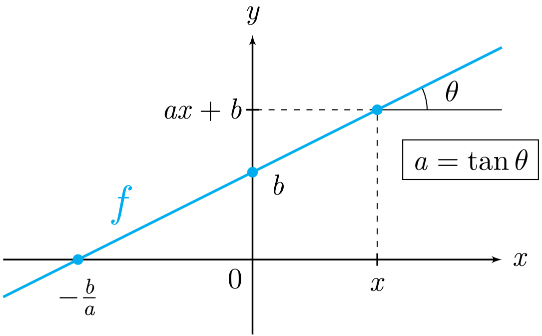
$D = \mathbb{R},\ C = \mathbb{R}$



Definition

A **linear function** $f\colon \mathbb{R} \to \mathbb{R}$ *is defined as*

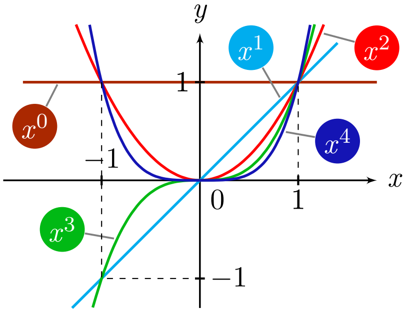
$$f(x) = ax + b, \qquad a \neq 0.$$

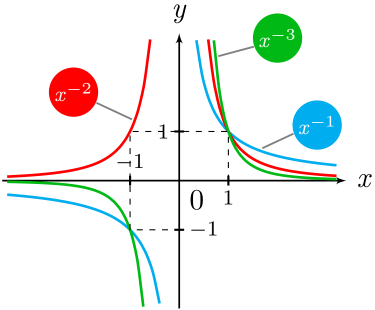


Definition

For every integer n we define

$$x^n = \begin{cases} 1 & \text{if } n = 0, \\ \underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ times}} & \text{if is } n \geq 1, \\ \dfrac{1}{x^{|n|}} & \text{if is } n < 0. \end{cases}$$

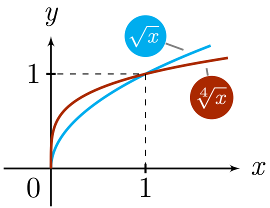




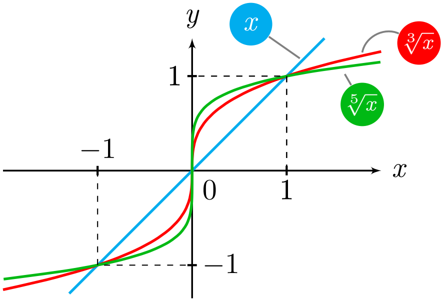
Definition

For every positive integer n we define the $\sqrt[n]{x} = x^{\frac{1}{n}}$ as the inverse of $f(x) = x^n$ where the domain of f is assumed to be

$$\begin{cases} [0,\infty) & \text{if } n \text{ is even,} \\ \mathbb{R} & \text{if } n \text{ is odd.} \end{cases}$$



$\sqrt{x} = \sqrt[2]{x}$



Definition

- *For arbitrary fractions* $\frac{p}{q}$ *(with* p *an integer and* q *a positive integer) we define*
$$x^{\frac{p}{q}} = \left(x^{\frac{1}{q}}\right)^p .$$
- *If* $\alpha \in \mathbb{R}$ *is not a fraction, then* x^α *is defined by limits. This is beyond the scope of this course.*

Basic properties

For arbitrary[1] x, y, α *and* β *we have*

1. $x^0 = 1$
2. $1^\alpha = 1$
3. $x^\alpha y^\alpha = (x\,y)^\alpha$
4. $x^{\alpha+\beta} = x^\alpha x^\beta$
5. $x^{\alpha-\beta} = \dfrac{x^\alpha}{x^\beta}$
6. $(x^\alpha)^\beta = x^{\alpha\beta}$

[1] Some combinations of x, y, α and β may not be defined.

- $$3^{1.1} \cdot 3^{0.7} = 3^{1.1+0.7} = 3^{1.8} = 3^{\frac{9}{5}} = \sqrt[5]{3^9}$$

- $$\frac{\left(\sqrt{11}\right)^3}{\sqrt{11}} = \left(\sqrt{11}\right)^{3-1} = \left(\sqrt{11}\right)^2 = 11$$

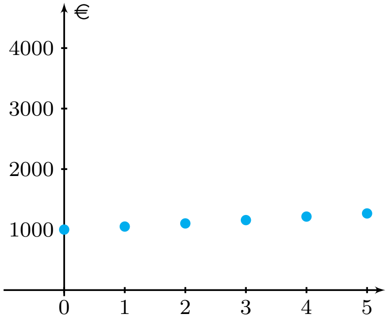
- $$\left(7^{\sqrt{2}}\right)^{\sqrt{2}} = 7^{\sqrt{2}\cdot\sqrt{2}} = 7^2 = 49$$

- $$7^\pi \cdot 8^\pi = (7 \cdot 8)^\pi = 56^\pi$$

- $$\left(\frac{4}{9}\right)^{\frac{1}{2}} = \frac{4^{\frac{1}{2}}}{9^{\frac{1}{2}}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3} \quad \text{or} \quad \left(\frac{4}{9}\right)^{\frac{1}{2}} = \sqrt{\frac{4}{9}} = \sqrt{\left(\frac{2}{3}\right)^2} = \frac{2}{3}$$

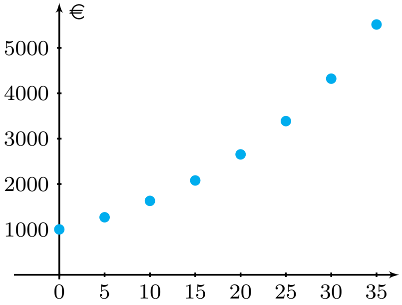
If I have 1000 Euro in a savings account and the bank gives 5% interest each year, what will be my savings after 5 years?

Year	Savings (€)	
0	1000	
1	$1000 \cdot (1.05)$	$= 1050.00$
2	$1000 \cdot (1.05)^2$	$= 1102.50$
3	$1000 \cdot (1.05)^3$	$= 1157.63$
4	$1000 \cdot (1.05)^4$	$= 1215.51$
5	$1000 \cdot (1.05)^5$	$= 1267.28$



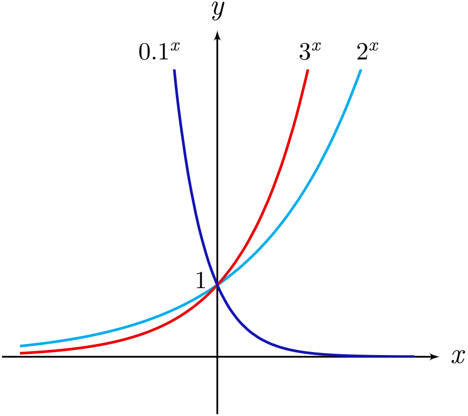
If I have 1000 Euro in a savings account and the bank gives 5% interest each year, what will be my savings after 35 years?

Year	Savings (€)	
0	1000	
5	$1000 \cdot (1.05)^{5}$	$= 1267.28$
10	$1000 \cdot (1.05)^{10}$	$= 1628.89$
15	$1000 \cdot (1.05)^{15}$	$= 2078.93$
20	$1000 \cdot (1.05)^{20}$	$= 2653.3$
25	$1000 \cdot (1.05)^{25}$	$= 3386.35$
30	$1000 \cdot (1.05)^{30}$	$= 4321.94$
35	$1000 \cdot (1.05)^{35}$	$= 5516.02$



Definition

*Let $a > 0$. The **exponential function** with base a is $f(x) = a^x$.*



$y = 2^x$

$y = 3^x$

$y = 10^{-x} = 0.1^x$

Exponential Functions.nb

Definition

- *If a quantity y depends on time and y is proportional to an exponential function, then we say that y* **grows exponentially**.
- *If the base is less than 1 we say that y* **decays exponentially**.

- the human population (annual growth percentage $\approx 1.14\%$),
- carbon dating (the half-life of ^{14}C is approximately 5730 years),
- compound interest,
- Moore's law: the number of transistors on integrated circuits doubles approximately every two years.

Exponential growth and decay

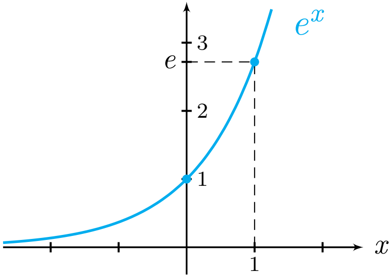
If y grows exponentially, then there are constants a and y_0 such that

$$y(x) = y_0 a^x.$$

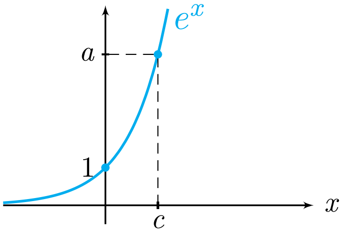
- The derivative of an exponential function is proportional to the function itself.
- If $f(x) = a^x$ then $f'(x) = K\, a^x$ for some constant K.
- There is one specific base value for which $K = 1$. This base is called e and is approximately

$$e \approx 2.71828182845904523536028747135266249775724709\ldots$$

- The function e^x is called the **natural exponential function**.



Exponential Functions.nb



- Let $a > 0$, then there is a constant $c \in \mathbb{R}$ such that
$$a = e^c.$$
- For every x the following holds:
$$a^x = (e^c)^x = e^{cx}$$

Exponential growth and decay

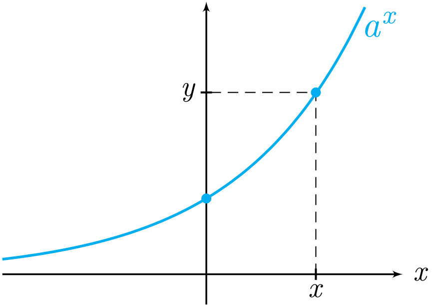
If y grows exponentially, then there are constants c and y_0 such that
$$y(x) = y_0 e^{cx}.$$

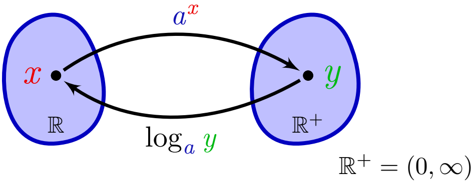
- If $c > 0$, then $a > 1$ hence y is exponentially growing, and c is called the **growth rate**.
- If $c < 0$, then $a < 1$ hence y is exponentially decaying, and c is called the **decay rate**.
- The constant y_0 is equal to $y(0)$, and is called the **initial value**.

Definition

The **logarithm with base** a *is the inverse of the exponential function with base* a*:*

$$y = a^x \qquad \Longleftrightarrow \qquad x = \log_a y$$





$\log_2 1 = 0$ because $2^0 = 1$,

$\log_2 2 = 1$ because $2^1 = 2$,

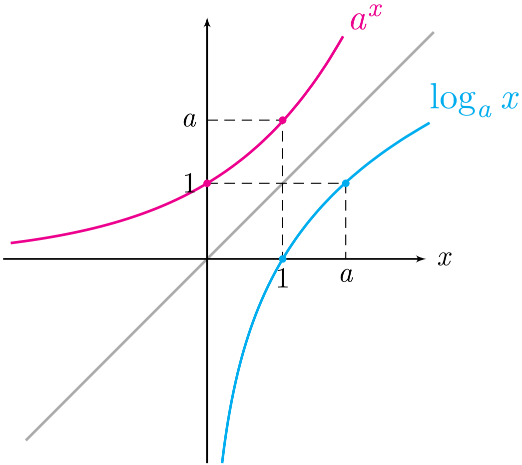
$\log_2 4 = 2$ because $2^2 = 4$,

$\log_{10} 1000 = 3$ because $10^3 = 1000$,

$\log_3 81 = 4$ because $3^4 = 81$,

$\log_9 81 = 2$ because $9^2 = 81$,

$\log_2 .25 = -2$ because $2^{-2} = \frac{1}{4} = .25$.



- The graph of $y = \log_a x$ is obtained by reflecting the graph of $y = a^x$ across the diagonal line $y = x$

Logarithmic laws

- $$\log_a 1 = 0$$
- $$\log_a a = 1$$
- $$\log_a (x\,y) = \log_a x + \log_a y$$
- $$\log_a \frac{x}{y} = \log_a x - \log_a y$$
- $$\log_a \frac{1}{y} = -\log_a y$$
- $$\log_a \left(x^p\right) = p \log_a x$$

- We write the logarithm with base 10 as $\boxed{\log x}$
- We write the logarithm with base e as $\boxed{\ln x}$
- The logarithm with base e is called the **natural logarithm**.